

THE ASYMPTOTIC DEVELOPMENT
FOR A CERTAIN
INTEGRAL FUNCTION OF
ZERO ORDER

BY

CHARLES W. COBB

ASSISTANT PROFESSOR OF MATHEMATICS
IN AMHERST COLLEGE

A PORTION OF A THESIS PRESENTED TO THE FACULTY OF THE
GRADUATE SCHOOL OF THE UNIVERSITY OF MICHIGAN
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

PRINTED AT

The Norwood Press

NORWOOD, MASS.

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1. Introduction. By means of the Maclaurin Sum Formula, Barnes* and Ford† have obtained asymptotic developments for certain integral functions of non-zero order, but for functions of zero order Barnes' analysis breaks down, while Ford has not treated the case, and Littlewood‡ holds that these functions should be studied by special methods. The present paper, however, uses the Maclaurin Sum Formula to obtain the asymptotic development for a typical function of zero order, and may therefore be considered as supplementing the work of Barnes and Ford, by showing that a common method of investigation, based on the Maclaurin formula, may be employed in determining asymptotic developments for integral functions of both orders.

2. The Maclaurin Sum Formula. The Maclaurin Sum Formula with remainder will be used in the following form: ||

If $f(w)$ is analytic throughout a vertical strip of the w complex plane, extending to an infinite distance above

* BARNES, E. W. P. *Phil. Trans. (A)* vol. 199, pp. 411-500; *Proc. Lond. Math. Soc.*, Series 2, vol. 3, pp. 273-295.

† FORD, W. B. *Annals of Math.*, second series, vol. 11 (1910).

‡ LITTLEWOOD, J. E. *Proc. Lond. Math. Soc.*, Series 2, vol. 5 (1907).

|| FORD, W. B. Lectures on Divergent Series given at Univ. of Michigan, 1910-1911.

and below the axis of reals and including the real points $w = a$, $w = b$, and is such that

$$\lim_{j \rightarrow +\infty} f(x \pm ij) e^{(\eta - \frac{2\pi}{h})j} = 0; \quad a \leq x \leq b$$

where η is some assignable positive quantity, we may write

$$\begin{aligned} \sum_{x=a}^{b-h} f(x) &= \frac{1}{h} \int_a^b f(x) dx - \frac{1}{2} [f(b) - f(a)] \\ &+ \frac{B_1 h}{2!} [f'(b) - f'(a)] + \dots \\ &+ \frac{(-1)^{m-1} B_m}{(2m)!} [f^{(2m-1)}(b) - f^{(2m-1)}(a)] + R_m \end{aligned}$$

where

$$R_m = \frac{(-1)^m}{(2m)! i h} \int_0^\infty \frac{[f^{(2m)}(x + i\theta y) - f^{(2m)}(x - i\theta y)]_{x=a}^{x=b}}{e^{2\pi \frac{y}{h}} - 1} \cdot y^{2m} dy$$

$$\begin{cases} \theta = 1 & \text{when } m = 0, \\ 0 < \theta < 1 & \text{when } m = 1, 2, 3 \dots \end{cases}$$

3. Problem. We shall apply this theorem to the discussion of a single type-function of zero order, though the method employed is evidently capable of broader application.

The problem is as follows:

Given the integral function of zero order,

$$F(z) = \prod_1^\infty \left(1 - \frac{z}{e^n}\right) = \exp. H(z); \quad z \text{ real or complex,}$$

it is proposed to consider the existence and determination of an asymptotic development for $H(z)$ in the

precise sense of "asymptotic" as originally formulated by Poincaré,* viz., a development of the form

$$H(z) = f(z) + \phi(z) \left[a_0 + \frac{a_1}{z} + \dots + \frac{a_{n-1}}{z^{n-1}} + \frac{a_n + \epsilon(z)}{z^n} \right];$$

$$\lim_{|z|=\infty} \epsilon(z) = 0; \quad n = 0, 1, 2 \dots.$$

This problem has been considered by other methods by Mellin,† Barnes,‡ Hardy,§ Mattson,|| and Littlewood.¶

So far as results are concerned, all agree as to the leading terms (terms that go to infinity as z becomes infinite), but no two results are exactly alike for the rest of the development. Such differences as present themselves are doubtless of form only and hence of minor importance, but the present treatment by means of the Maclaurin Sum Formula tends to unify the problem of determining asymptotic developments for integral functions, by reducing cases of both non-zero and zero order to a common method of treatment.

4. From our definition, § 3,

$$(1) \quad H(z) = \lim_{x \rightarrow \infty} \left[\sum_{x=1}^{x-1} \log(e^x - z) - \sum_{x=1}^{x-1} x \right]$$

and by the logarithm of a complex number Z , say, we shall mean: $\log Z = \log R + i\Phi$;

$$Z = R(\cos \Phi + i \sin \Phi); \quad -2\pi < \Phi \leq 0.$$

In order to evaluate $H(z)$ for large z , we first regard z as fixed and place $\frac{e^p}{z} = \theta$; p = greatest integer in

* *Acta Math.*, vol. 8 (1886), pp. 295-344.

† *Acta Soc. Fennicæ*, t. XXIX. § *Quarterly Journal of Math.* (1905).

‡ *Phil. Trans. (A)* vol. 199. || *Thèse*, Upsal (1905).

¶ *Proc. Lond. Math. Soc.*, Series 2, vol. 5 (1907).

$\log |z|$; $\frac{1}{e} < |\theta| < 1$. Inasmuch as the following analysis holds only when $\frac{1}{e} < |\theta| < 1$, z may not lie on any circle whose center is at the origin and whose radius is e^p ($p = 1, 2, 3 \dots$) but z may be chosen anywhere else in the complex plane.

To evaluate $\sum_{x=a}^{b-h} f(x)$, the theorem of § 2 requires that the corresponding $f(w)$ shall be analytic throughout a vertical strip of the w -plane, including the points $w = a$, $w = b$. So we write

$$(2) \sum_{x=1}^{x-1} \log (e^x - z) = \sum_{x=1}^{p-1} \log (e^x - z) + \log (e^p - z) \\ + \sum_{x=p+1}^{x-1} \log (e^x - z)$$

and observe that the function $f(w) = \log (e^w - z)$ is analytic in the vertical strip containing the points $w = 1$, $w = p$, and also in the strip containing the points $w = p+1$, $w = x$.

We have then, putting $m = 0$, $h = 1$,

$$\sum_{x=1}^{p-1} \log (e^x - z) = \int_1^p \log (e^x - z) dx - \frac{1}{2} [\log (e^p - z) \\ - \log (e - z)] + \Omega_z(p) - \Omega_z(1)$$

where

$$\Omega_z(x) = -i \int_0^x [\log (e^{x+iy} - z) - \log (e^{x-iy} - z)] \cdot \frac{dy}{e^{2\pi y} - 1}.$$

Also,

$$\sum_{x=p+1}^{x-1} \log (e^x - z) = \int_{p+1}^x \log (e^x - z) dx - \frac{1}{2} [\log (e^x - z) \\ - \log (e^{p+1} - z)] + \Omega_z(x) - \Omega_z(p+1).$$

Taking the terms of the right-hand members in the above order,

$$\int_1^p \log (e^x - z) dx = \left[x \log (e^x - z) - \frac{x^2}{2} - \int \frac{xz}{e^x - z} dx \right]_{x=1}^{x=p}$$

$$= p \log (e^p - z) - \frac{p^2}{2} - \log (e - z) + \frac{1}{2} - \int_1^p \frac{xz}{e^x - z} dx.$$

Now,

$$- \int_1^p \frac{xz}{e^x - z} dx = \int_1^p \frac{xz}{z - e^x} dx$$

$$= \int_1^p x \left(1 + \frac{e^x}{z} + \left(\frac{e^x}{z} \right)^2 + \dots \right) dx.$$

The power series in $\frac{e^x}{z}$ is uniformly convergent for all x , $1 \leq x \leq p$, since in this interval $|e^x| < |z|$. Hence, integrating term by term,

$$- \int_1^p \frac{xz}{e^x - z} dx$$

$$= \left[\frac{x^2}{2} + \frac{e^x}{z}(x-1) + \frac{1}{4} \left(\frac{e^x}{z} \right)^2 (2x-1) + \frac{1}{9} \left(\frac{e^x}{z} \right)^3 (3x-1) + \dots \right]_{x=1}^{x=p}$$

$$= \frac{p^2}{2} - \frac{1}{2} + S_1 + S_1'$$

where

$$S_1 = \frac{e^p}{z}(p-1) + \frac{1}{4} \left(\frac{e^p}{z} \right)^2 (2p-1) + \frac{1}{9} \left(\frac{e^p}{z} \right)^3 (3p-1) + \dots,$$

$$S_1' = -\frac{1}{4} \left(\frac{e}{z} \right)^2 - \frac{2}{9} \left(\frac{e}{z} \right)^3 - \dots.$$

Collecting terms,

$$(3) \quad \log (e^p - z) + \sum_{x=1}^{p-1} \log (e^x - z) = (p + \frac{1}{2}) \log (e^p - z)$$

$$- \log (e - z) + S_1 + S_1' + \Omega_z(p) - \Omega_z(1).$$

Consider next the terms arising from

$$\sum_{x=p+1}^{x-1} \log (e^x - z).$$

We have

$$\int_{p+1}^x \log (e^x - z) dx = \left[x \log (e^x - z) - \frac{x^2}{2} - \int_{x=p+1}^x \frac{xz}{e^x - z} dx \right].$$

Now

$$-\int_{p+1}^x \frac{xz}{e^x - z} dx = -\int_{p+1}^x \left[\frac{xz}{e^x} \left(1 + \frac{z}{e^x} + \left(\frac{z}{e^x} \right)^2 + \dots \right) \right] dx;$$

$$|z| < e^{x'}$$

$$= \left[\frac{z}{e^x} (x+1) + \frac{1}{4} \left(\frac{z}{e^x} \right)^2 (2x+1) + \frac{1}{9} \left(\frac{z}{e^x} \right)^3 (3x+1) + \dots \right]_{x=p+1}^{x=x}$$

$$= -(p+2) \left(\frac{z}{e^{p+1}} \right) - \frac{2p+3}{4} \left(\frac{z}{e^{p+1}} \right)^2 - \frac{3p+4}{9} \left(\frac{z}{e^{p+1}} \right)^3 - \dots$$

if we neglect terms that go to zero exponentially when x is infinite. Calling the power series in $\left(\frac{z}{e^{p+1}} \right)$, S_2 , we may write

$$(4) \quad \sum_{x=p+1}^{x-1} \log (e^x - z) = x \log (e^x - z) - \frac{x^2}{2} - (p+1) \log (e^{p+1} - z)$$

$$+ \frac{(p+1)^2}{2} - \frac{1}{2} \log (e^x - z) + \frac{1}{2} \log (e^{p+1} - z) + S_2 + \Omega_z(x)$$

$$- \Omega_z(p+1).$$

Noting that $\sum_{x=1}^{x-1} x = \frac{x^2 - x}{2}$ and that

$$\lim_{x \rightarrow \infty} \left[\left(x - \frac{1}{2} \right) \log (e^x - z) - \frac{x^2}{2} - \frac{x^2 - x}{2} \right] = 0, \text{ we have}$$

$$(5) \quad H(z) = (p + \frac{1}{2}) \log (e^p - z) - \frac{1}{2} \log (e - z)$$

$$- \left(p + \frac{1}{2} \right) \log (e^{p+1} - z) + \frac{(p+1)^2}{2}$$

$$+ S_1 + S_1' + S_2 + \Omega_z(x) - \Omega_z(p+1)$$

$$+ \Omega_z(p) - \Omega_z(1).$$

To evaluate S_1 and S_2 , note that since each series is absolutely convergent, we may write, setting

$$\frac{e^p}{z} = \theta, \frac{z}{e^{p+1}} = \frac{1}{e\theta}; \quad \frac{1}{e} < |\theta| < 1,$$

$$S_1 = p [-\log(1 - \theta)] + \int_0^\theta \frac{\log(1 - r)}{r} dr$$

$$S_2 = (p + 1) \log \left(1 - \frac{1}{e\theta}\right) + \int_0^{\frac{1}{e\theta}} \frac{\log(1 - r)}{r} dr,$$

for,
$$\frac{\log(1 - r)}{r} = -1 - \frac{r}{2} - \frac{r^2}{3} - \dots$$

and
$$S_1 = \theta(p - 1) + \frac{\theta^2}{4}(2p - 1) + \frac{\theta^3}{9}(3p - 1) + \dots,$$

$$\begin{aligned} S_2 &= -(p + 2)\left(\frac{1}{e\theta}\right) - \frac{2p + 3}{4}\left(\frac{1}{e\theta}\right)^2 - \frac{3p + 4}{9}\left(\frac{1}{e\theta}\right)^3 - \dots \\ &= (p + 1)\left[-\left(\frac{1}{e\theta}\right) - \frac{1}{2}\left(\frac{1}{e\theta}\right)^2 - \frac{1}{3}\left(\frac{1}{e\theta}\right)^3 - \dots\right] \\ &\quad - \left(\frac{1}{e\theta}\right) - \frac{1}{4}\left(\frac{1}{e\theta}\right)^2 - \frac{1}{9}\left(\frac{1}{e\theta}\right)^3 - \dots \end{aligned}$$

From the definition of θ , $p = \log z + \log \theta$ and hence (as will be shown presently) for large z ,

$$\begin{aligned} (6) \quad H(z) &= \frac{1}{2}(\log z)^2 + \log z \left[-\frac{1}{2} + \log(-1)\right] \\ &\quad + \log \left(1 - \frac{e}{z}\right) + e(\theta) + \left\{ \frac{a}{z_1} + \dots + \frac{a_n + \epsilon(z)}{z^n} \right\}; \quad \lim_{z \rightarrow \infty} \epsilon(z) = 0 \end{aligned}$$

where $e(\theta)$ is a constant depending only on θ .^{*} Since θ is by definition a periodic function of z , provided z moves along a straight line passing through the origin, the same is true for $e(\theta)$.

It remains to establish equation (6) and to evaluate $e(\theta)$ and the a 's.

^{*} Cf. LITTLEWOOD, *loc. cit.*, 397 (13).

We have from (5), after the substitution

$$p = \log z + \log \theta,$$

$$\begin{aligned} (7) \quad H(z) &= (\log z + \log \theta + \tfrac{1}{2})[\log z + \log (\theta - 1)] \\ &- \tfrac{1}{2} \log (e - z) - (\log z + \log \theta + \tfrac{1}{2})[\log z + \log (e\theta - 1)] \\ &+ \frac{(\log z + \log \theta + 1)^2}{2} \\ &+ (\log z + \log \theta)[\log (e\theta - 1) - 1 - \log \theta - \log (1 - \theta)] \\ &+ \log \left(1 - \frac{1}{e\theta}\right) + \left[\int_0^{\frac{1}{e\theta}} dr + \int_0^\theta dr\right] \left(\frac{\log (1-r)}{r}\right) \\ &+ S_1' + \Omega_z(x) - \Omega_z(p+1) + \Omega_z(p) - \Omega_z(1). \end{aligned}$$

Consider now

$$\Omega(1) \equiv -i \int_0^\infty \log \frac{e^{1+iy} - z}{e^{1-iy} - z} \cdot \frac{dy}{e^{2y} - 1}.$$

We may write

$$\begin{aligned} &\log \left[\left(1 - \frac{e^{1+iy}}{z}\right)(-z) \right] - \log \left[\left(1 - \frac{e^{1-iy}}{z}\right)(-z) \right] \\ &= \log \left(1 - \frac{e^{1+iy}}{z}\right) - \log \left(1 - \frac{e^{1-iy}}{z}\right) \\ &= -\frac{e^{1+iy}}{z} - \frac{e^{2(1+iy)}}{2z^2} - \frac{e^{3(1+iy)}}{3z^3} - \dots \\ &\quad + \frac{e^{1-iy}}{z} + \frac{e^{2(1-iy)}}{2z^2} + \frac{e^{3(1-iy)}}{3z^3} + \dots \\ &= -\frac{e}{z} (\cos y + i \sin y) - \frac{1}{2} \left(\frac{e}{z}\right)^2 (\cos 2y + i \sin 2y) - \dots \\ &\quad + \frac{e}{z} (\cos y - i \sin y) + \frac{1}{2} \left(\frac{e}{z}\right)^2 (\cos 2y - i \sin 2y) + \dots \\ &= -2i \left[\frac{e}{z} (\sin y) + \frac{1}{2} \left(\frac{e}{z}\right)^2 \sin 2y + \frac{1}{3} \left(\frac{e}{z}\right)^3 \sin 3y + \dots \right] \end{aligned}$$

Moreover, we have the relation *

$$\int_0^\infty \frac{\sin px}{e^{qx} - 1} dx = \frac{\pi}{2q} \cdot \frac{e^{\frac{2p\pi}{q}} + 1}{e^{\frac{2p\pi}{q}} - 1} - \frac{1}{2p}.$$

Hence, integrating term by term,

$$\begin{aligned} -\Omega_z(1) &= \frac{e}{z} \left(\frac{1}{2} \frac{e+1}{e-1} - 1 \right) + \frac{1}{2} \left(\frac{e}{z} \right)^2 \left(\frac{1}{2} \frac{e^2+1}{e^2-1} - \frac{1}{2} \right) \\ &+ \frac{1}{3} \left(\frac{e}{z} \right)^3 \left(\frac{1}{2} \frac{e^3+1}{e^3-1} - \frac{1}{3} \right) + \dots \end{aligned}$$

We have a right, in the preceding, to integrate term by term,† for, put

$$\frac{1}{y} \left[\frac{e}{z} \sin y + \frac{1}{2} \left(\frac{e}{z} \right)^2 \sin 2y + \dots \right] \equiv \sum f_n y$$

and

$$\frac{y}{e^{2\pi y} - 1} \equiv \phi(y)$$

then, (a) when $y = 0$, $\sum f_n(y)$ is $\frac{0}{0}$ but approaches a definite limit $\frac{e}{z} + \left(\frac{e}{z} \right)^2 + \left(\frac{e}{z} \right)^3 + \dots$ as y goes to zero.

In determining this limit, we have the right to differentiate term by term since both the sine series and the derived cosine series are uniformly convergent for all y 's,

as $\left| \frac{e}{z} \right| < 1$.

(b) When $y \neq 0$, $\sum f_n(y)$ is uniformly convergent for all y 's and $\int_0^\infty |\phi(y)| dy$ is convergent, hence we may integrate term by term.

* BIERENS DE HAAN, *Table of Definite Integrals*, 264, 2.

† (f. BROMWICH, *Infinite Series*, § 176 (A).

Again, consider

$$\Omega_z(p) \equiv -i \int_0^\infty \log \frac{e^{p+iy} - z}{e^{p-iy} - z} \cdot \frac{dy}{e^{2\pi y} - 1}$$

We may write

$$\begin{aligned} \log \left[\left(1 - \frac{e^{p+iy}}{z} \right) (-z) \right] - \log \left[\left(1 - \frac{e^{p-iy}}{z} \right) (-z) \right] \\ = \log \left(1 - \frac{e^{p+iy}}{z} \right) - \log \left(1 - \frac{e^{p-iy}}{z} \right). \end{aligned}$$

Putting $\frac{e^p}{z} = \theta$, and proceeding as in the case of $\Omega_z(1)$,

we have

$$\begin{aligned} \Omega_z(p) = - \left[\theta \left(\frac{1}{2} \frac{e+1}{e-1} - 1 \right) + \frac{1}{2} \theta^2 \left(\frac{1}{2} \frac{e^2+1}{e^2-1} - \frac{1}{2} \right) \right. \\ \left. + \frac{1}{3} \theta^3 \left(\frac{1}{2} \frac{e^3+1}{e^3-1} - \frac{1}{3} \right) + \dots \right] \end{aligned}$$

and the power series in θ is convergent since $|\theta| < 1$.

Consider

$$\Omega_z(p+1) \equiv -i \int_0^\infty \log \frac{e^{p+1+iy} - z}{e^{p+1-iy} - z} \cdot \frac{dy}{e^{2\pi y} - 1}.$$

We may write

$$\begin{aligned} \log \left[\left(1 - \frac{z}{e^{p+1+iy}} \right) e^{p+1+iy} \right] - \log \left[\left(1 - \frac{z}{e^{p+1-iy}} \right) e^{p+1-iy} \right] \\ = \log \left(1 - \frac{z}{e^{p+1+iy}} \right) - \log \left(1 - \frac{z}{e^{p+1-iy}} \right) + 2iy. \end{aligned}$$

Putting $\frac{z}{e^{p+1}} = \frac{1}{e\theta}$, and proceeding as above, we have

$$\begin{aligned} -\Omega_z(p+1) = -\frac{1}{12} - \left[\frac{1}{e\theta} \left(\frac{1}{2} \frac{e+1}{e-1} - 1 \right) \right. \\ \left. + \frac{1}{2} \left(\frac{1}{e\theta} \right)^2 \left(\frac{1}{2} \frac{e^2+1}{e^2-1} - \frac{1}{2} \right) + \frac{1}{3} \left(\frac{1}{e\theta} \right)^3 \left(\frac{1}{2} \frac{e^3+1}{e^3-1} - \frac{1}{3} \right) + \dots \right], \end{aligned}$$

and the power series in $\frac{1}{e\theta}$ is convergent since $\left| \frac{1}{e\theta} \right| < 1$.

In the same way

$$\Omega_z(x) = \frac{1}{12} + \left[\frac{z}{e^x} \left(\frac{1}{2} \frac{e+1}{e-1} - 1 \right) + \frac{1}{2} \left(\frac{z}{e^x} \right)^2 \left(\frac{1}{2} \frac{e^2+1}{e^2-1} - \frac{1}{2} \right) \right. \\ \left. + \frac{1}{3} \left(\frac{z}{e^x} \right)^3 \left(\frac{1}{2} \frac{e^3+1}{e^3-1} - \frac{1}{3} \right) + \dots \right]$$

and

$$\lim_{x \rightarrow \infty} \Omega_z(x) = \frac{1}{12}.$$

Certain further reductions may be made, viz.,

$$-\Omega_z(1) + S_1' \equiv \log \left(1 - \frac{e}{z} \right) + \sum_1^{\infty} \frac{1}{z} \frac{e^n + 1}{2n} \left(\frac{e}{z} \right)^n.$$

Also,

$$\Omega_z(p) + \int_0^{\theta} \frac{\log(1-r)}{r} dr = - \sum_1^{\infty} \frac{1}{z} \frac{e^n + 1}{2n} \cdot \frac{e^n + 1}{e^n - 1} \theta^n$$

and

$$-\Omega_z(p+1) + \Omega_z(x) + \int_0^{\frac{1}{e\theta}} \frac{\log(1-r)}{r} dr \\ = - \sum_1^{\infty} \frac{1}{z} \frac{e^n + 1}{2n} \cdot \frac{e^n + 1}{e^n - 1} \left(\frac{1}{e\theta} \right)^n.$$

We have, then, finally,

$$(8) \quad H(z) = \frac{1}{2}(\log z)^2 + \log z \left[-\frac{1}{2} + \log(-1) \right] \\ + \log \left(1 - \frac{e}{z} \right) + e(\theta) \\ + \left\{ \frac{a_1}{z} + \dots + \frac{a_{n-1}}{z^{n-1}} + \frac{a_n + \epsilon(z)}{z^n} \right\}; \quad \lim_{z \rightarrow \infty} \epsilon(z) = 0$$

where

$$e(\theta) = -\frac{1}{2}(\log \theta)^2 + \frac{1}{2} \log(\theta - 1) + \frac{1}{2} \log(e\theta - 1) \\ + \log \theta [-1 + \log(-1)] - \frac{1}{2} - \sum_1^{\infty} \frac{1}{z} \frac{e^n + 1}{2n} \cdot \frac{e^n + 1}{e^n - 1} \theta^n \\ - \sum_1^{\infty} \frac{1}{z} \frac{e^n + 1}{2n} \cdot \frac{e^n + 1}{e^n - 1} \left(\frac{1}{e\theta} \right)^n,$$

and $\frac{a_k}{z^k} = \frac{1}{2k} \cdot \frac{e^k + 1}{e^k - 1} \left(\frac{e}{z}\right)^k$; $k = 1, 2, \dots, n-1$.

As the power series $\sum_1^{\infty} \frac{1}{2n} \frac{e^n + 1}{e^n - 1} \left(\frac{e}{z}\right)^n$ is convergent, evidently $\lim_{z=\infty} \epsilon(z) = 0$, and the problem of § 3 is completely solved.

